CP1 & honors (not AP®)

Uncertainty & Error Analysis

Unit: Laboratory & Measurement

NGSS Standards/MA Curriculum Frameworks (2016): SP4

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): SP3.C

Mastery Objective(s): (Students will be able to…)

- Determine the uncertainty of a measured or calculated value.

Success Criteria:

- Take analog measurements to one extra digit of precision.
- Correctly estimate measurement uncertainty.
- Correctly read and interpret stated uncertainty values.
- Correctly propagate uncertainty through calculations involving addition/subtraction and multiplication/division.

Language Objectives:

- Understand and correctly use the terms “uncertainty” and “relative error.”
- Correctly explain the process of estimating and propagating uncertainty.

Tier 2 Vocabulary: uncertainty, error

Notes:

In science, unlike mathematics, there is no such thing as an exact answer. Ultimately, every quantity is limited by the precision and accuracy of the measurements that it came from. If you can only measure a quantity to within 10 %, that means any calculation that is derived from that measurement can’t be any better than ±10 %.

Error analysis is the practice of determining and communicating the causes and extents of uncertainty in your results. Error analysis involves understanding and following the uncertainty in your data, from the initial measurements to the final calculated and reported results.

Note that the word “error” in science has a different meaning from the word “error” in everyday language. In science, **“error” means “uncertainty.”** If you reported that you drive (2.4 ± 0.1) miles to school every day, you would say that this distance has an error of ±0.1 mile. This does not mean your car’s odometer is wrong; it means that the actual distance *could be* 0.1 mile more or 0.1 mile less—*i.e.,* somewhere between 2.3 and 2.5 miles. ***When you are analyzing your results, <u>never</u> use the word “error” to mean mistakes that you might have made!***

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Uncertainty

The uncertainty or error of a measurement describes how close the actual value is likely to be to the measured value. For example, if a length was measured to be 22.3 cm, and the uncertainty was 0.3 cm (meaning that the measurement is only known to within ± 0.3 cm), we could represent this measurement in either of two ways:

22.3 ± 0.3 cm* 22.3(3) cm

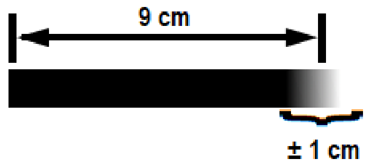
The first of these states the variation (±) explicitly in cm (the actual unit). The second shows the variation in the last digits shown.

What it means is that the true length is approximately 22.3 cm, and is statistically likely† to be somewhere between 22.0 cm and 22.6 cm.

Absolute Uncertainty (Absolute Error)

Absolute uncertainty (or absolute error) refers to the uncertainty in the actual measurement.

For example, consider the rectangle below (not to scale):



The length of this rectangle is approximately 9 cm, but the exact length is uncertain because we can't determine exactly where the right edge is.

We would express the measurement as 9 ± 1 cm, because the right edge could be different from where we marked it by up to 1 cm in either direction. The ± 1 cm of uncertainty is called the *absolute error*.

Every measurement has a limit to its precision, based on the method used to measure it. This means that ***every measurement has uncertainty***.

* The unit is assumed to apply to both the value and the uncertainty. It would be more pedantically correct to write (9 ± 1) cm, but this is rarely done. The unit for the value and uncertainty should be the same. For example, a value of 10.63 m ± 2 cm should be rewritten as 10.63 ± 0.02 m

† Statistically, the standard uncertainty is one standard deviation, which is discussed on page 61.

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Relative Uncertainty (Relative Error)

Relative uncertainty (usually called relative error) shows the error or uncertainty as a fraction of the measurement.

The formula for relative error is $\text{R.E.} = \frac{\text{uncertainty}}{\text{measured value}}$

For example, the rectangle in the example above had a measurement of 9 ± 1 cm. We can think of this as an uncertainty of "1 cm out of 9 cm". In math, the phrase "out of" means "divide," so we would represent this as 1 cm ÷ 9 cm. However, with algebra, it is always best to write division as a fraction, so we would write this as:

$$\frac{1\ \cancel{\text{cm}}}{9\ \cancel{\text{cm}}} = \frac{1}{9} = 0.111$$

Notice that the units cancel. Relative error is a *dimensionless* quantity, meaning that it has no dimensions (and therefore no units*).

The relative error is simply the fraction (usually expressed as a decimal) of the measurement that is uncertain.

Percent Error

Percent error is simply the relative error expressed as a percentage. You can turn relative error into percent error by multiplying by 100.

In the example above, the relative error of 0.111 would be 11.1 % error.

* Dimensions and units are not quite the same thing. A dimension is what a quantity represents, such as length. A unit is a specific increment used to measure that dimension, such as meters or centimeters.

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Uncertainty of A Single Measurement

If you have the ability to measure a quantity that is not changing (such as the mass or length of an object), you will get the same value every time you measure it. This means you have only one data point.

When you have only one data point, the uncertainty is the limit of how well you can measure it. This will be your best educated guess, based on how closely you think you actually measured the quantity. This means you need to take measurements as carefully and precisely as possible, because *every careless measurement needlessly increases the uncertainty of the result*.

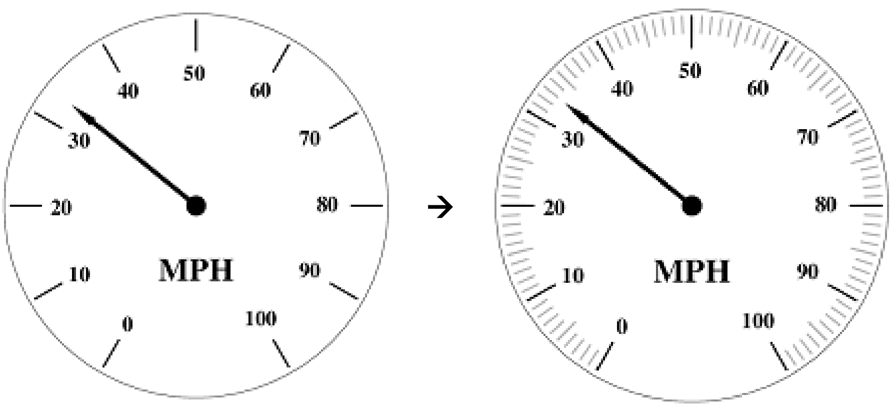
Digital Measurements

For digital equipment, if the reading is *stable* (not changing), look up the published precision of the instrument in its user's manual. (For example, many balances used in high schools have a readability of 0.01 g but are only precise to within ± 0.02 g.) If there is no published value (or the manual is not available), assume the uncertainty is ± 1 in the last digit.

If the reading is *unstable* (changing), state the reading as the average of the highest and lowest values, and the uncertainty as half of the range: (highest – lowest)/2, which is the amount that you would need to add to or subtract from the average to obtain either of the extremes. (However, the uncertainty can never be less than the published uncertainty of the equipment).

Analog Measurements

When making analog measurements, *always* estimate one extra digit beyond the finest markings on the equipment. For example, if you saw the speedometer on the left, you would imagine that each tick mark was divided into ten smaller tick marks like the one on the right.



what you see:
between 30 & 40 MPH

what you visualize:
33 ± 1 MPH

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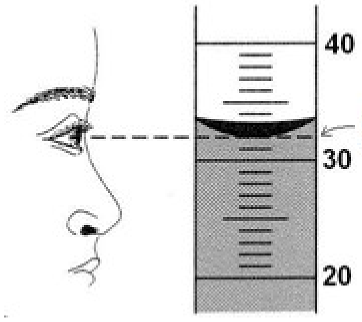
Note that the ***measurement and uncertainty must be expressed to the same decimal place***.

For example, in the diagram below, the graduated cylinder is marked in 1 mL increments. When measuring volume in this graduated cylinder*, you would estimate and write down the volume to the nearest 0.1 mL, as shown:



In the above experiment, you *must* record the volume as:

32.0 ± 0.1 mL ← correct

32 ± 0.1 mL ← wrong

32 ± 1 mL ← inadequate

In other words, the zero at the end of 32.0 mL is *required*. It is necessary to show that *you measured the volume to the nearest tenth, not to the nearest one.*

When estimating, the uncertainty depends on how well you can see the markings, but you can usually assume that the estimated digit has an uncertainty of $\pm\frac{1}{10}$ of the finest markings on the equipment. Here are some examples:

Equipment	Typical Markings	Estimate To	Assumed Uncertainty
ruler	1 mm	0.1 mm	± 0.1 mm
25 mL graduated cylinder	0.2 mL	0.02 mL	± 0.02 mL
thermometer	1 °C	0.1 °C	± 0.1 °C

* Remember that for most liquids, which have a downward meniscus, volume is measured at the *bottom* of the meniscus.

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Calculating the Uncertainty of a *Set* of Measurements

When you have measurements of multiple separately-generated data points, the uncertainty is calculated using statistics, so that some specific percentage of the measurements will fall within the average, plus or minus the uncertainty.

Note that statistical calculations are beyond the scope of this course. This information is provided for students who have taken (or are taking) a statistics course and are interested in how statistics are applied to uncertainty.

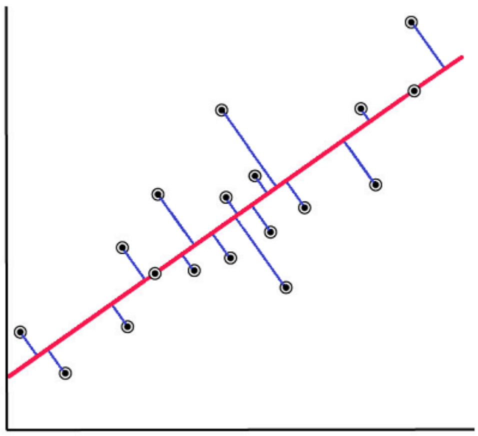
Ten or More Independent Measurements: Standard Deviation

If you have a large enough set of independent measurements (at least 10), then the uncertainty is the standard deviation of the mean. (Independent measurements means you set up the situation that generated the data point on separate occasions. *E.g.,* if you were measuring the length of the dashes that separate lanes on a highway, independent measurements would mean measuring different lines that were likely generated by different line painting apparatus. Measuring the same line ten times would not be considered independent measurements.)

standard deviation (σ): the average of how far each data point is from its expected value.

The standard deviation is calculated mathematically as the average difference between each data point and the value predicted by the best-fit line (see Graphical Solutions & Linearization on page 77).

best-fit line: a line that represents the expected value of your responding variable for values of your manipulated variable. The best-fit line minimizes the total accumulated error (difference between each actual data point and the line).



A small standard deviation means that most or all of the data points lie close to the best-fit line. A larger standard deviation means that on average, the data points lie farther from the line.

Unless otherwise stated, ***the standard deviation is the uncertainty (the "plus or minus") of a calculated quantity***. *E.g.,* a measurement of 25.0 cm with a standard deviation of 0.5 cm would be expressed as (25.0 ± 0.5) cm.

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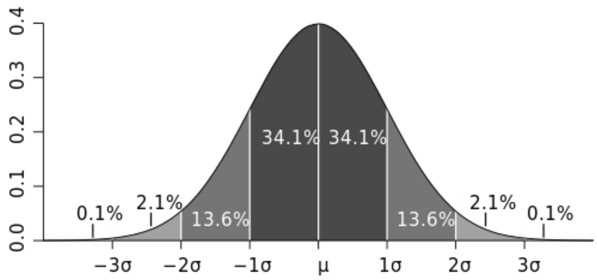
<u>correlation coëfficient</u> (*R* or R^2 value): a measure of how linear the data are—how well they approximate a straight line. In general, an R^2 value of less than 0.9 means that the data are not linear, there was a problem with one or more data points, or there was a problem with the entire experiment.

The expected distribution of values relative to the mean is called the Gaussian distribution (named after the German mathematician Carl Friedrich Gauss.) It looks like a bell, and is often called a "bell curve".

Statistically, approximately two-thirds (actually 68.2 %) of the measurements are expected to fall within one standard deviation of the mean, *i.e.,* within the standard uncertainty.



There is an equation for standard deviation, though most people don't use the equation because they calculate the standard deviation using the statistics functions on a calculator or computer program.

However, note that most calculators and statistics programs calculate the *sample* standard deviation (σ_s), whereas the uncertainty should be the standard deviation *of the mean* (σ_m). This means:

$$u = \sigma_m = \frac{\sigma_s}{\sqrt{n}}$$

and:

$$\text{reported value} = \bar{x} \pm u = \bar{x} \pm \sigma_m = \bar{x} \pm \frac{\sigma_s}{\sqrt{n}}$$

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Fewer than Ten Independent Measurements

While the standard deviation of the mean is the correct approach when we have a sufficient number of data points, often we have too few data points (small values of *n*), which causes the calculated standard deviation to predict a much larger uncertainty than we probably actually have.

If you have only a few independent measurements (fewer than 10), then you have too few data points for the standard deviation to represent the uncertainty. In this case, we can estimate the standard uncertainty by finding the range and dividing by two.*

Example:

Suppose you let a toy car go down a ramp and across the floor until it stopped, and the distances were 3.1 m, 2.9 m, and 3.3 m. The average of these distances is 2.9 m, and we can see by inspection that if we add 0.2 m we would get the farthest distance, and if we subtract 0.2 m we would get the shortest, so we would express the distance as 3.1 ± 0.2 m.

If we needed to calculate the uncertainty for a less convenient set of numbers, we would find the range and divide it by 2. In the above example, the range is 3.3 – 2.9 = 0.4 m. If we divide the range by 2, we get 0.2 m as expected.

This also works for a single measurement that is drifting. For example, suppose you measured a mass on a balance and the reading drifted between 3.46 g and 3.58 g:

$$\bar{x} = \frac{3.46 + 3.58}{2} = 3.52$$

$$range = 3.58 - 3.46 = 0.12$$

$$u \approx \frac{range}{2} \approx \frac{0.12}{2} \approx 0.06$$

You would record the balance reading as 3.52 ± 0.06 g.

* Some texts suggest dividing by $\sqrt{3}$ instead of dividing by 2. For so few data points, the distinction is not important enough to add another source of confusion for students.

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Propagating Uncertainty in Calculations

When you perform calculations using numbers that have uncertainty, you need to propagate the uncertainty through the calculation.

Crank Three Times

The simplest (to understand) way to calculate uncertainty is the “crank three times” method. The “crank three times” method involves:

1. Perform the calculation using the actual numbers. This gives the result (the part before the ± symbol).

2. Perform the calculation a second time, using the end of the range for each value that would give the *smallest* result. (Note that with fractions, this means you need to subtract the uncertainty for values in the numerator and add the uncertainty for values in the denominator.) This gives the lower limit of the range.

3. Perform the calculation a third time using the end of the range for each value that would give the *largest* result. This gives the upper limit of the range.

4. Assuming you have fewer than ten data points, use the approximation that the uncertainty = $u \approx \frac{range}{2}$.

The advantage to “crank three times” is that it’s easy to understand and you are therefore less likely to make a mistake. The disadvantage is that it can become unwieldy when you have multi-step calculations.

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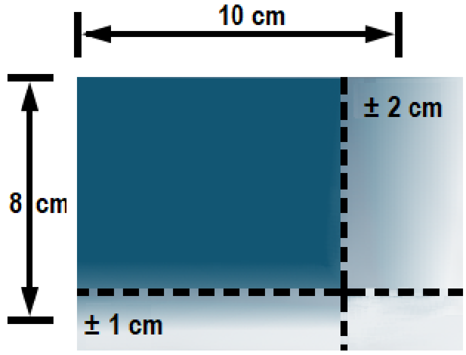
Relative Error

When you have a series of calculations, the "crank three times" method takes a lot of effort. Using relative error is much easier, because you only need to calculate the relative errors of each of the measurements, add them together, and multiply by your final result to get the (absolute) uncertainty, in the same units.

For example, consider the following rectangle. (Note that it is deliberately uncertain exactly where the bottom and right edges of the rectangle are.)



The base (length) of the rectangle is 10 ± 2 cm, and the height (width) is 8 ± 1 cm. This means that the area is approximately 80 cm^2.

We could calculate the uncertainty by adding the area of the uncertain part of the base (the vertical section at the right), which is 1 × 10 = 10 cm^2, and the area of the uncertain part of the height (the horizontal section at the bottom), which is 2 × 8 = 16 cm^2. The total uncertainty is therefore 10 + 16 = 26 cm^2. (In this case we double-count the overlap, because it's uncertain ***both*** because of the uncertainty in the base ***and*** because of the uncertainty of the height.)

However, it is much easier (both conceptually and mathematically) to use relative error.

The fraction of the length that is uncertain (the relative error of the length) is $\frac{2\ \text{cm}}{10\ \text{cm}} = 0.2$. The fraction of the width that is uncertain (the relative error of the width) is $\frac{1\ \text{cm}}{8\ \text{cm}} = 0.125$.

Note that relative error is dimensionless (does not have any units), because the numerator and denominator have the same units, which means the units cancel.

If we add these relative errors together, we get 0.2 + 0.125 = 0.325, which is the total relative error for the entire rectangle (length and width combined).

If we multiply this total relative error of 0.325 by the area of the rectangle (80 cm^2), we get the uncertainty for the area: (0.325)(80 cm^2) = ± 26 cm^2.

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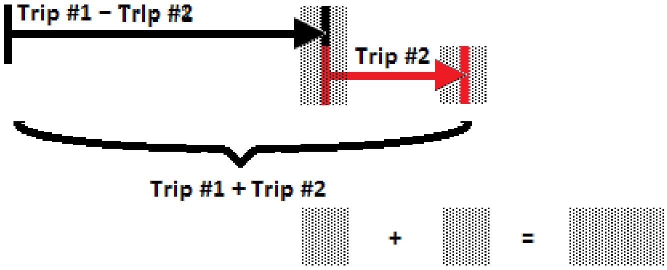
Addition & Subtraction: Add the *Absolute* Errors

When quantities with uncertainties are added or subtracted, the uncertainties have the same units, which means we can simply add the quantities to get the answer, and then just add the uncertainties to get the total uncertainty. Note that this only works for addition & subtraction, because the units are the same.

If the calculation involves ***addition*** or ***subtraction***, add the *absolute* errors.

Imagine you walked for a distance and measured it. That measurement has some uncertainty. Then imagine that you started from where you stopped and walked a second distance and measured it. The second measurement also has uncertainty. The total distance is the distance for Trip #1 + Trip #2.



Because there is uncertainty in the distance of Trip #1 and *also* uncertainty in the distance of Trip #2, it is easy to see that the total uncertainty when the two trips are added together is the sum of the two uncertainties.

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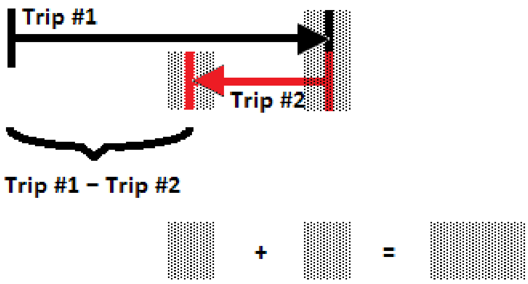
Now imagine that you walked for a distance and measured it, but then you turned around and walked back toward your starting point for a second distance and measured that. Again, both measurements have uncertainty.



Notice that, even though the distances are subtracted to get the answer, the *uncertainties still accumulate*. As before, the uncertainty in where Trip #1 ended becomes the uncertainty in where Trip #2 started. There is also uncertainty in where Trip #2 ended, so again, the total uncertainty is the *sum* of the two uncertainties.

For a numeric example, consider the problem:

$$(8.45 \pm 0.15\text{ cm}) - (5.43 \pm 0.12\text{ cm})$$

Rewriting in column format:

$$\begin{array}{rl} & 8.45 \pm 0.15\text{ cm} \\ - & \underline{5.43 \pm 0.12\text{ cm}} \\ & \boxed{3.02 \pm 0.27\text{ cm}} \end{array}$$

Notice that even though we had to subtract to find the answer, we had to *add the uncertainties*.

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Multiplication & Division: Add the *Relative* Errors

If the calculation involves ***multiplication or division***, we can't just add the uncertainties (absolute errors), because the units do not match. Therefore, we need to add the *relative* errors to get the total relative error, and then convert the relative error back to absolute error afterwards.

Note: ***Most of the calculations that you will perform in physics involve multiplication and/or division, which means almost all of your uncertainty calculations throughout the course will use relative error.***

For example, if we have the problem (2.50 ± 0.15 kg) × $(0.30 \pm 0.06\ \frac{\text{m}}{\text{s}^2})$, we would do the following:

1. **Calculate the result** using the equation.

   $F_{net} = ma$

   $F_{net} = (2.50)(0.30) = 0.75\ \text{N}$ ← Result

2. **Calculate the relative error for each of the measurements**:

   The relative error of (2.50 ± 0.15) kg is $\frac{0.15\ \cancel{\text{kg}}}{2.50\ \cancel{\text{kg}}} = 0.06$

   The relative error of $(0.30 \pm 0.06)\ \frac{\text{m}}{\text{s}^2}$ is $\frac{0.06\ \cancel{\frac{\text{m}}{\text{s}^2}}}{0.30\ \cancel{\frac{\text{m}}{\text{s}^2}}} = 0.20$

   (Notice that the units cancel.)

3. **Add the relative errors** to find the total relative error:

   0.06 + 0.20 = 0.26 ← Total Relative Error

4. **Multiply the total relative error** (step 3) **by the result** (from step 1 above) to convert the uncertainty back to the correct units.

   (0.26)(0.75 N) = 0.195 N

   (Notice that the units come from the result.)

5. **Combine the result with its uncertainty** and round appropriately:

   $F_{net} = 0.75 \pm 0.195\ \text{N}$

   Because the uncertainty is specified, the answer is technically correct without rounding, but it is good form to round uncertainties to the appropriate number of significant figures, and *round the result to the same decimal place*:

   $F_{net} = 0.75 \pm 0.20\ \text{N}$

For the kinds of experiments you will do in physics class, it is usually sufficient to show the uncertainty to one or two significant figures, and then round the answer to the same place value.

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Exponents

Calculations that involve ***exponents*** use the same rule as for multiplication and division. If you think of exponents as multiplying a number by itself the indicated number of times, it means you would need to add the relative error of that number that many times.

In other words, when a value is raised to an exponent, multiply its relative error by the exponent.

Note that this applies even when the exponent is a fraction (meaning roots). For example:

A ball is dropped from a height of 1.8 ± 0.2 m and falls with an acceleration of $9.81 \pm 0.02\,\frac{\text{m}}{\text{s}^2}$. You want to find the time it takes to fall, using the equation $t = \sqrt{\frac{2a}{d}}$. Because $\sqrt{x}$ can be written as $x^{\frac{1}{2}}$, the equation can be rewritten as $t = \frac{\sqrt{2a}}{\sqrt{d}} = \frac{(2a)^{\frac{1}{2}}}{d^{\frac{1}{2}}}$

Using the steps on the previous page:

1. The result is $t = \sqrt{\frac{2a}{d}} = \sqrt{\frac{2(9.81)}{1.8}} = \sqrt{10.9} = 3.30\,\text{s}$
2. The relative errors are:

   distance: $\frac{0.2\,\text{m}}{1.8\,\text{m}} = 0.111$

   acceleration: $\frac{0.02\,\frac{\text{m}}{\text{s}^2}}{9.81\,\frac{\text{m}}{\text{s}^2}} = 0.0020$

3. Because of the square roots in the equation, the total relative error is: $\frac{1}{2}(0.111) + \frac{1}{2}(0.002) = 0.057$
4. The absolute uncertainty for the time is therefore (3.30)(0.057) = ± 0.19 s.
5. The answer is therefore 3.30 ± 0.19 s. However, we have only one significant figure of uncertainty for the height, so it would be better to round to 3.3 ± 0.2 s.

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Summary of Uncertainty Calculations

Uncertainty of a Single Quantity

Measured Once

Make your best educated guess of the uncertainty based on how precisely you were able to measure the quantity and the uncertainty of the instrument(s) that you used.

Measured Multiple Times (Independently)

- If you have a lot of data points, the uncertainty is the standard deviation of the mean, which you can get from a calculator that has statistics functions.
- If you have few data points, use the approximation $u \approx \frac{r}{2}$.

Uncertainty of a Calculated Value

Calculations Use Only Addition & Subtraction

The uncertainties all have the same units, so just add the uncertainties of each of the measurements. The total is the uncertainty of the result.

Calculations Use Multiplication & Division (and possibly Exponents)

The uncertainties don't all have the same units, so you need to use relative error.:

1. Perform the desired calculation. (Answer the question without worrying about the uncertainty.)
2. Find the relative error of each measurement. $\text{R.E.} = \frac{\text{uncertainty } (\pm)}{\text{measured value}}$
3. If the equation includes an exponent (including roots, which are fractional exponents), multiply each relative error by its exponent in the equation.
4. Add the relative errors to find the total relative error.
5. Multiply the total relative error from step 4 by the answer from step 1 to get the absolute uncertainty (±) in the correct units.
6. If desired, round the uncertainty to the appropriate number of significant digits and round the answer to the same place value.

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Homework Problems

Because the answers are provided, you must show sufficient work in order to receive credit.

1. **(M = Must Do)** In a 4 × 100 m relay race, the four runners' times were: (10.52 ± 0.02) s, (10.61 ± 0.01) s, (10.44 ± 0.03) s, and (10.21 ± 0.02) s. What was the team's (total) time for the event, including the uncertainty?

   Answer: 41.78 ± 0.08 s

2. **(S = Should Do)** After school, you drove a friend home and then went back to your house. According to your car's odometer, you drove 3.4 miles to your friend's house (going past your house on the way). Then you drove 1.2 miles back to your house. If the uncertainty in your car's odometer reading is 0.1 mile, how far is it from school directly to your house (including the uncertainty)?

   Answer: 2.2 ± 0.2 mi.

3. **(M = Must Do)** A baseball pitcher threw a baseball for a distance of (18.44 ± 0.05) m in (0.52 ± 0.02) s.

   a. What was the velocity of the baseball in meters per second? (*Divide the distance in meters by the time in seconds.*)

      Answer: $35.46\,\frac{\text{m}}{\text{s}}$

   b. What are the relative errors of the distance and time? What is the total relative error?

      Answer: distance: 0.0027; time: 0.0385; total R.E.: 0.0412

   c. Calculate the uncertainty of the velocity of the baseball and express your answer as the velocity (from part a above) plus or minus the uncertainty that you just calculated, with correct rounding.

      Answer: $35.46 \pm 1.46\,\frac{\text{m}}{\text{s}}$ which rounds to $35 \pm 1\,\frac{\text{m}}{\text{s}}$

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4. **(S)** A rock that has a mass of 8.15 ± 0.25 kg is sitting on the top of a cliff that is 27.3 ± 1.1 m high. What is the gravitational potential energy of the rock (including the uncertainty)? The equation for this problem is $U_g = mgh$. In this equation, *g* is the acceleration due to gravity on Earth, which is equal to $9.81 \pm 0.02\,\frac{m}{s^2}$, and the unit for energy is J (joules).

Answer: 2 183 ± 159 J

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5. **(S)** You drive West on the Mass Pike, from Route 128 to the New York state border, a distance of 127 miles. The EZ Pass transponder determines that your car took 1 hour and 54 minutes (1.9 hours) to complete the trip, and you received a ticket in the mail for driving $66.8\,\frac{\text{mi.}}{\text{hr.}}$ in a $65\,\frac{\text{mi.}}{\text{hr.}}$ zone. The uncertainty in the distance is ± 1 mile and the uncertainty in the time is ± 30 seconds (± 0.0083 hours). Can you use this argument to fight the ticket and win? (You can win if you prove that because of the uncertainty, your speed *could* have been less than $65\,\frac{\text{mi.}}{\text{hr.}}$.)

Answer: No, this argument won't work. Your average speed is $66.8 \pm 0.8\,\frac{\text{mi.}}{\text{hr.}}$. Therefore, the minimum that your speed could have been is $66.8 - 0.8 = 66.0\,\frac{\text{mi.}}{\text{hr.}}$.

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