honors & AP®

Graphical Solutions & Linearization

Unit: Laboratory & Measurement

NGSS Standards/MA Curriculum Frameworks (2016): SP4, SP5

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.B, 2.A, 2.B, 2.D, 3.C

Mastery Objective(s): (Students will be able to…)

- Use a graph to calculate the relationship between two variables.

Success Criteria:

- Graph has the manipulated variable on the *x*-axis and the responding variable on the *y*-axis.
- Graph includes best-fit line that appears to minimize the total accumulated distance between the points and the line.
- Axes and best-fit line drawn with straightedge.
- Divisions on axes are evenly spaced.
- Slope of line determined correctly (rise/run).
- Slope used correctly in calculation of desired result.

Language Objectives:

- Explain why a best-fit line gives a better answer than calculating an average.
- Explain how the slope of the line relates to the desired quantity.

Tier 2 Vocabulary: plot, axes

Summary of Concepts & Equations:5

linearization: rearranging an equation so that it would produce a graph that is a straight line, *i.e.,* in $y = mx + b$ form.

Notes:

Most experiments in a high-school physics class involve finding a mathematical relationship between two quantities. While it is possible to simply measure each quantity once and calculate, or to measure several times and calculate the average, an approach that measures the relationship across a range of values will provide a better result.

Use this space for summary and/or additional notes:

honors & AP®

A common way to achieve this is to manipulate equations and plot data such that the expected result is a straight line (*i.e.*, in $y = mx + b$ form), and the experimental quantity is calculated by plotting a best-fit line and determining the slope. This means that you need to either:

1. Plot an ***accurate*** graph on graph paper: or using a computer or calculator. If you use graph paper:
    - Carefully plot each data point as close to its actual location as you are capable of.
    - The best-fit line ***must be drawn with a straightedge*** (such as a ruler).
    - Position the straightedge so that the line you will draw with it minimizes the total accumulated distance from the line to each data point.
    - Calculate the slope using the actual rise and run between two points that lie on the best-fit line. (Do not use your data points unless they actually lie on the best-fit line.)
2. Use a calculator or a statistics app on a computer:
    - Enter the x and y values into the calculator or app (such as Desmos).
    - Have the calculator or app perform a *linear regression*.
    - The results of the linear regression will be:
        - The slope of the line. *This quantity or its reciprocal is the quantity that you need.*
        - The y-intercept
        - The correlation coefficient.

    See *Calculating the Uncertainty of a Set of Measurements* starting on page 62 for an explanation of these quantities.

Use this space for summary and/or additional notes:

honors & AP®

For example, suppose you wanted to calculate the spring constant of a spring by stretching it and measuring the resulting force applied by the spring. (This will be covered in the *Springs* topic, starting on page 348.) You obtain the following data:

Displacement (m)	0	0.05	0.10	0.15	0.20	0.25	0.30
Spring Force (N)	0	0.9	1.7	2.7	4.1	5.1	5.8

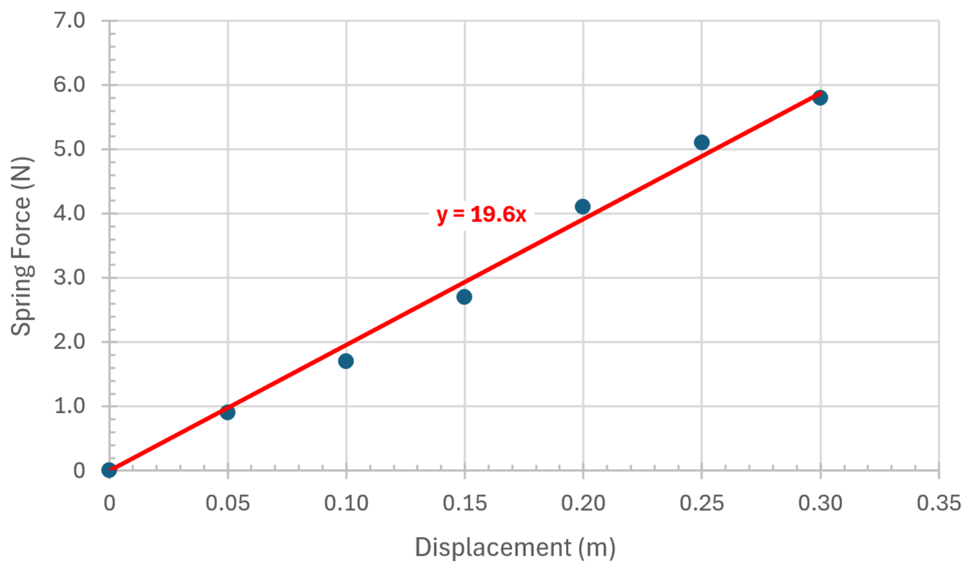
The relevant equation is Hooke's Law, $F_s = kx$. Note that Hooke's Law is already in $y = mx + b$ form:

$$\begin{array}{ccccccc} y & = & m & x & + & b \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ F_s & = & k & x & + & 0 \end{array}$$

In our equation:

- F_s corresponds to y, so we will plot F_s (force) on the y-axis.
- x corresponds to x, so we will plot x (displacement) on the x-axis. ☺
- k corresponds to m (the slope), so the slope of our graph will be the spring constant k. (Recall that this is the quantity that we want.)

The plot looks like the following:



Conveniently, the spreadsheet that was used to plot the best-fit line (trendline) is able to display the equation for the line. The slope is 19.6, which means our spring constant is $19.6\,\frac{\text{N}}{\text{m}}$.

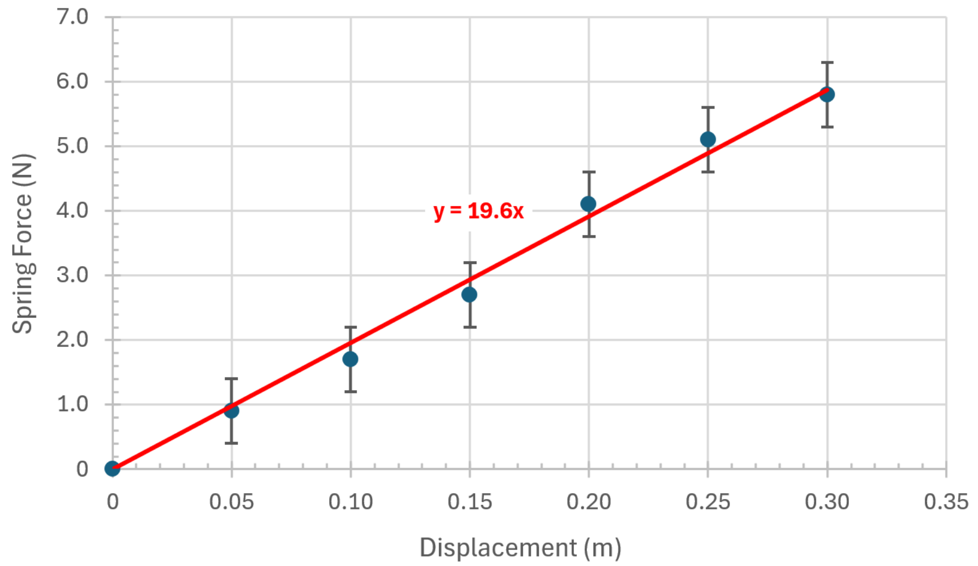
Use this space for summary and/or additional notes:

honors (not AP®)

Uncertainty and Best-Fit lines

If you determine a quantity by linearizing, the best-fit line should pass within the uncertainty of every data point. (This is, after all, what uncertainty means!) You can show/check this by plotting the graph with error bars, which show the maximum and minimum values for each data point, based on the uncertainty.

Suppose our force measurements each had an uncertainty of ± 0.5 N. The best-fit line with error bars would look like this:



The top of each error bar is the force plus the uncertainty, which is the maximum possible value (assuming we have estimated our uncertainty appropriately). The bottom of each error bar is the force minus the uncertainty, which is the minimum possible value.

Notice that the best fit line passes through all of the error bars. This is important. ***If the best-fit line calculated by linear regression does not pass through the error bars, the equation of the line must be manipulated until it does.*** (This is, after all, what uncertainty means!)

If it is not possible to plot a line that passes through all of the error bars, this suggests that either:

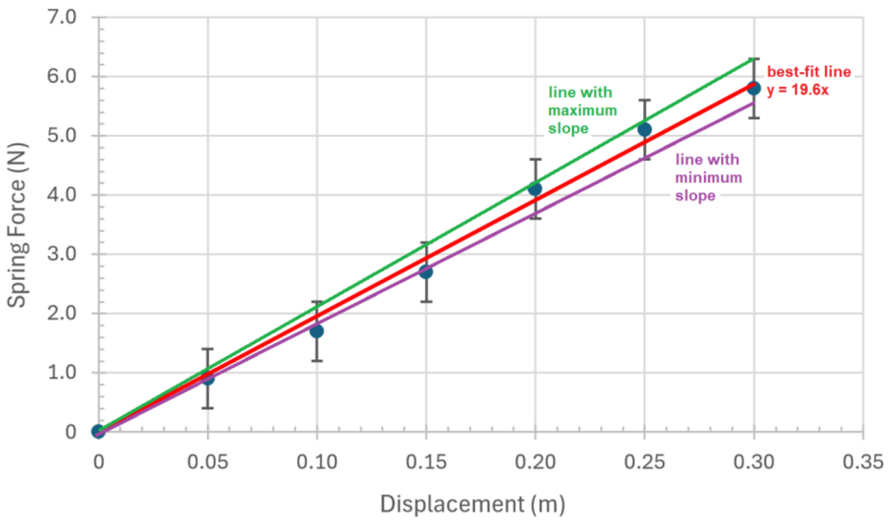
- The uncertainty was underestimated.
- The data points whose error bars do not intersect the line may be outliers.

If a data point is an outlier, you should attempt to determine the cause—it was most likely an unidentified problem with that data point. (If there is a problem with your results, the first thing you should check is your procedure.)

Use this space for summary and/or additional notes:

honors (not AP®)

Calculating the uncertainty of a value that was calculated from a best-fit line is difficult. The largest possible value is the line with maximum slope that passes through all of the error bars (shown in green below). The smallest possible value is the line with minimum slope that passes through all of the error bars (shown in purple).



(Note that because zero force must result in zero displacement, the intercept of all of the lines was forced through zero.)

Manipulating the equations of the two "worst-fit" lines is tedious, and beyond the scope of a high school course.

A Simple Approximation of the Uncertainty of a Best-Fit Line

In this course, you will simply use relative error to propagate your uncertainties, as described in the *Uncertainty & Error Analysis* topic, starting on page 57.

Choose the data point with the largest relative error for each of your measured quantities, and calculate the total relative error and absolute error of the calculated result from those. Note that the largest relative error for each quantity may come from different data points.

Use this space for summary and/or additional notes:

honors & AP®

Linearization

Often, it is desirable to use linear regression (the process of calculating the best-fit line) in situations where the equation itself is non-linear. This looks confusing at first, but it's actually simple.

All you need to do is rearrange the equation so that your manipulated (independent) variable and the quantity you want to calculate are both on one side of the equation, and your responding (dependent) variable is on the other.

For example, suppose you want to determine the electrical current (I) passing through a circuit. You know the total electrical resistance of the circuit (*R*), and you are able to measure the power consumption (*P*).

The equation relating these quantities is $P = I^2R$. In slope-intercept form, this looks like:

$$\begin{array}{ccccccc} y & = & m & x & + & b \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ P & = & I^2 & R & + & 0 \end{array}$$

This means you should plot a graph of *P* vs. *R*. You should force the intercept of the best-fit line through zero, and the slope will be I^2. Once you determine the slope, you need to take the square root of it to get the value of I.

Suppose instead that you had the same electrical circuit, but you were able to measure power (*P*) and current (I), and you wanted to determine the resistance (*R*). The equation is still $P = I^2R$, which we can rewrite as $P = RI^2$. Now, the slope-intercept form of our equation is:

$$\begin{array}{ccccccc} y & = & m & x & + & b \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ P & = & R & I^2 & + & 0 \end{array}$$

This time, we need to plot a graph of *P* vs. I^2. Again, you should force the best-fit line through zero, and the slope will be *R*.

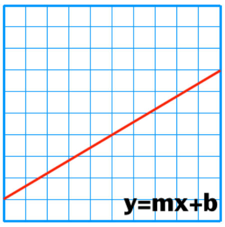
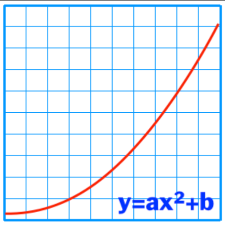
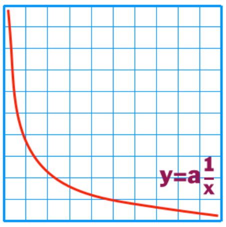
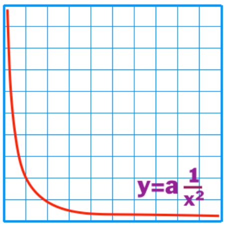
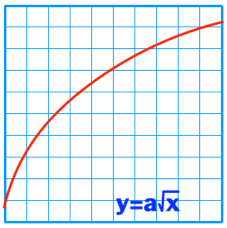
Use this space for summary and/or additional notes:

honors & AP®

Recognizing Shapes of Graphs

When you know the equation in advance, it is easy to rearrange the equation in order to linearize it. However, if you do not know the equation before looking at the data, you can often make a guess based on its shape.

It is useful to memorize the general shapes of the graphs* of some common equations, so you can predict the type of relationship and the form of the equation. Note, however, that some graphs look similar to others. Just because a graph "looks like" it fits a particular equation doesn't necessarily mean that the equation is correct!

Plot of *y vs. x*	Equation	Linear Plot
	**Linear** $y = mx + b$ b = y-intercept	y vs. x slope = m
	**Power** $y = ax^2$ or $y = ax^2 + b$ b = minimum y-value	y vs. x^2 slope = a
	**Inverse** $y = \frac{a}{x}$ or $y = a \cdot \frac{1}{x}$ undefined (∞) at $x = 0$	y vs. $\frac{1}{x}$ slope = a
	**Inverse Square** $y = \frac{a}{x^2}$ or $y = a \cdot \frac{1}{x^2}$ undefined (∞) at $x = 0$	y vs. $\frac{1}{x^2}$ slope = a
	**Square Root** $y = a\sqrt{x}$	y vs. $\sqrt{x}$ slope = a

*Graphs by Tony Wayne. Used with permission.

Use this space for summary and/or additional notes:

Homework Problems

In each of the following problems, you want to find the desired quantity graphically by plotting a best-fit line and finding the slope.

1. You want to find the acceleration (*a*) of an object. You measured velocity (*v*) and time (*t*).

Desired Quantity	Equation	Constant (Known)	Measured
a	$v = at$ *	—	v, t

   a. Which quantities would you plot on the x- and y-axes to cause the equation to be in $y = mx + b$ form?

   b. What is the expression for the slope?

   c. How would you calculate the desired quantity (*a*) from the expression for the slope?

2. You want to find the kinetic energy (*K*) of an object. You measured mass (*m*) and velocity (*v*).

Desired Quantity	Equation	Constant (Known)	Measured
K	$K = \frac{1}{2}mv^2$	—	m, v

   a. Which quantities would you plot on the x- and y-axes to cause the equation to be in $y = mx + b$ form?

   b. What is the expression for the slope?

   c. How would you calculate the desired quantity (*K*) from the expression for the slope?

* The equation is actually $v - v_o = at$. Because the object starts from rest, $(v_o = 0)$, which means v_o drops out of the equation.

Use this space for summary and/or additional notes:

3. You want to find the gravitational force (F_g) between two objects. You measured the mass of one of the objects (m_2) and the distance between them (r).

Desired Quantity	Equation	Constant (Known)	Measured
F_g	$F_g = \frac{Gm_1m_2}{r^2}$	G, m_1	m_2, r

a. Which quantities would you plot on the x- and y-axes to cause the equation to be in $y = mx + b$ form?

b. What is the expression for the slope?

c. How would you calculate the desired quantity (F_g) from the expression for the slope?

4. You want to find the centripetal acceleration (a_c) of an object. You measured the object's velocity (v) and the radius of the circle that it is moving in (r).

Desired Quantity	Equation	Constant (Known)	Measured
a_c	$a_c = \frac{v^2}{r}$	—	v, r

a. Which quantities would you plot on the x- and y-axes to cause the equation to be in $y = mx + b$ form?

b. What is the expression for the slope?

c. How would you calculate the desired quantity (a_c) from the expression for the slope?

Use this space for summary and/or additional notes: