Vector Multiplication

Unit: Laboratory & Mathematics

NGSS Standards/MA Curriculum Frameworks (2016): SP5

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.1.B.1

Mastery Objective(s): (Students will be able to…)

- Correctly use and interpret the symbols "•" and "×" when multiplying vectors.
- Finding the dot product & cross product of two vectors.

Success Criteria:

- Magnitudes and directions are correct.

Language Objectives:

- Explain how to interpret the symbols "•" and "×" when multiplying vectors.

Tier 2 Vocabulary: magnitude, direction, dot, cross

Notes:

With scalar (ordinary) numbers, there is only one way to multiply them, which you learned in elementary school. Vectors, however, can be multiplied in three different ways.

dot product: multiplication of two vectors that results in a scalar.

$$\vec{\boldsymbol{A}} \bullet \vec{\boldsymbol{B}} = C$$

cross product: multiplication of two vectors that results in a new vector.

$$\vec{\boldsymbol{I}} \times \vec{\boldsymbol{J}} = \vec{\boldsymbol{K}}$$

tensor product: multiplication of two vectors that results in a tensor. $\vec{\boldsymbol{A}} \otimes \vec{\boldsymbol{B}}$ is a matrix of vectors that results from multiplying the respective components of each of the two vectors. It describes the effect of each component of the vector on each component of every other vector in the array. Tensors are beyond the scope of a high school physics course.

Multiplying a Vector by a Scalar

Multiplying a vector by a scalar is like multiplying a variable by a number. The magnitude changes, but the direction does not. For example, in physics, displacement equals velocity times time:

$$\vec{\boldsymbol{d}} = \vec{\boldsymbol{v}}t$$

Velocity is a vector; time is a scalar. The magnitude is the velocity times the time, and the direction of the displacement is the same as the direction of the velocity.

If the two vectors have opposite directions, the equation needs a negative sign. For example, the force applied by a spring equals the spring constant (a scalar quantity) times the displacement:

$$\vec{\boldsymbol{F}}_s = -k\vec{\boldsymbol{x}}$$

The negative sign in the equation signifies that the force applied by the spring is in the opposite direction from the displacement.

The Dot (Scalar) Product of Two Vectors

The scalar product of two vectors is called the "dot product". Dot product multiplication of vectors is represented with a dot:

$$\vec{\boldsymbol{A}} \bullet \vec{\boldsymbol{B}}$$ *

The dot product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is:

$$\vec{\boldsymbol{A}} \bullet \vec{\boldsymbol{B}} = AB\cos\theta$$

where A is the magnitude of $\vec{\boldsymbol{A}}$, B is the magnitude of $\vec{\boldsymbol{B}}$, and θ is the angle between the two vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$.

For example, in physics, *work* (a scalar quantity) is the dot product of the vectors *force* and *displacement* (distance):

$$W = \vec{\boldsymbol{F}} \bullet \vec{\boldsymbol{d}} = Fd\cos\theta$$

* pronounced "A dot B"

The Cross (Vector) Product of Two Vectors

The vector product of two vectors is called the cross product. Cross product multiplication of vectors is represented with a multiplication sign:

$$\vec{A} \times \vec{B}$$ [*]

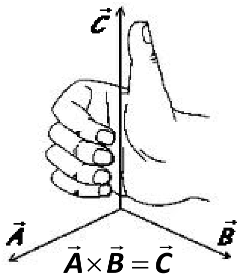
The magnitude of the cross product of vectors $\vec{A}$ and $\vec{B}$ that have an angle of θ between them is given by the formula:

$$\vec{A} \times \vec{B} = AB\sin\theta$$

The direction of the cross product is a little difficult to make sense out of. You can figure it out using the "right hand rule":

Position your right hand so that your fingers curl from the first vector to the second. Your thumb points in the direction of the resultant vector.

Note that this means that the resultant vectors for $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ point in *opposite* directions, *i.e.,* the cross product of two vectors is <u>*not*</u> commutative!



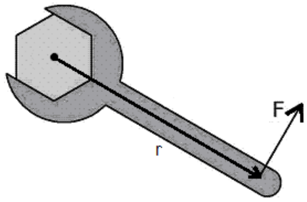
$$\vec{A} \times \vec{B} = \vec{C}$$
$$\vec{B} \times \vec{A} = -\vec{C}$$

On a two-dimensional piece of paper, a vector coming toward you (out of the page) is denoted by a set of ⊙ ⊙ ⊙ ⊙ ⊙ symbols, and a vector going away from you (into the page) is denoted by a set of ⊗ ⊗ ⊗ ⊗ ⊗ symbols.

Think of these symbols as representing an arrow inside a tube or pipe. The dot represents the tip of the arrow coming toward you, and the "X" represents the fletches (feathers) on the tail of the arrow going away from you.)

[*] pronounced "A cross B"

In physics, torque is a vector quantity that is derived by a cross product.



The torque produced by a force $\vec{F}$ acting at a radius $\vec{r}$ is given by the equation:

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta$$

Because the direction of the force is usually perpendicular to the displacement, it is usually true that $\sin\theta = \sin 90° = 1$. This means the magnitude $rF\ \sin\theta = rF(1) = rF$. Using the right-hand rule, we determine that the *direction* of the resultant torque vector is coming out of the page.

(The force generated by the interaction between charges and magnetic fields, a topic covered in AP® Physics 2, is also a cross product.)

Thus, if you are tightening or loosening a nut or bolt that has right-handed (standard) thread, the torque vector will be in the direction that the nut or bolt moves.

Vector Jokes

Now that you understand vectors, here are some bad vector jokes:

Q: What do you get when you cross an elephant with a bunch of grapes?

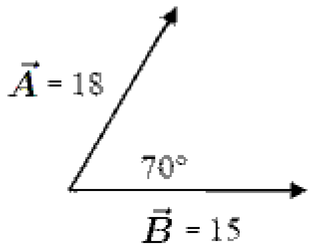
A:   $\sin\theta$

Q: What do you get when you cross an elephant with a mountain climber?

A: You can't do that! A mountain climber is a scalar ("scaler," meaning someone who scales a mountain).

Homework Problems

For the following vectors $\vec{A}$ & $\vec{B}$:



1. **(M)** Determine $\vec{A}\bullet\vec{B}$

2. **(M)** Determine $\vec{A}\times\vec{B}$ (both magnitude and direction)