Uncertainty & Error Analysis

Unit: Laboratory & Measurement

NGSS Standards: N/A

MA Curriculum Frameworks (2006): N/A

AP Physics 2 Learning Objectives: SP 5.2, SP 5.3

Knowledge/Understanding:

- Understand what uncertainty of measurement means.

Skills:

- Read and interpret uncertainty values.
- Estimate measurement errors.
- Propagate estimate of error/uncertainty through calculations.

Language Objectives:

- Understand and correctly use the terms "uncertainty," "standard uncertainty," and "relative error."
- Accurately describe and apply the concepts described in this section using appropriate academic language.

Notes:

In science, unlike mathematics, there is no such thing as an exact answer. Ultimately, every quantity is limited by the precision and accuracy of the measurements that it came from. If you can only measure a quantity to within 10%, that means any number that is derived from that measurement can't be any better than ±10%.

Error analysis is the practice of determining and communicating the causes and extents of possible errors or uncertainty in your results. Error analysis involves understanding and following the uncertainty in your data from the initial measurement to the final report.

Use this space for summary and/or additional notes:

Uncertainty

The uncertainty of a measurement describes how different the correct value is likely to be from the measured value. For example, if a length was measured to be 22.336 cm, and the uncertainty was 0.030 cm (meaning that the measurement is only known to within ±0.030 cm), we could represent this measurement in either of two ways:

(22.336 ± 0.030) cm   22.336(30) cm

The first of these states the variation (±) explicitly in cm (the actual unit). The second is shows the variation in the last digits shown.

Note that scientists use the word "error" to mean uncertainty. This is different from the vernacular usage of "error" to mean a mistake. When you discuss "sources of error" in an experiment, you are explaining the uncertainty in your results, and where it may have come from. ***"Sources of error" does not mean "Potential mistakes you might have made."*** This is similar to the difference between scientific use of the word "theory" ("a model that explains every observation that has ever been made") as opposed to vernacular use ("my own guess about what I think might be going on").

Absolute Uncertainty

Absolute uncertainty (or absolute error) refers to the uncertainty in the actual measurement, such as (3.64 ± 0.22) cm.

Use this space for summary and/or additional notes:

Relative Error

Relative error (or relative uncertainty) shows the error or uncertainty as a fraction of the measurement. (Percent error, which you used in chemistry last year, is related to relative error.)

The formula for relative error is $\text{R.E.} = \dfrac{\text{uncertainty}}{\text{measured value}}$

For example, a measurement of (3.64 ± 0.22) cm would have a relative error of 0.22 out of 3.64. Mathematically, we express this as:

$$\text{R.E.} = \frac{0.22}{3.64} = 0.0604$$

To turn relative error into percent error, multiply by 100. A relative error of 0.0604 is the same as 6.04% error.

Note that relative error does not have any units. This is because the numerator and denominator have the same units, so the units cancel.

To turn relative error back to absolute error, multiply the relative error by the measurement.

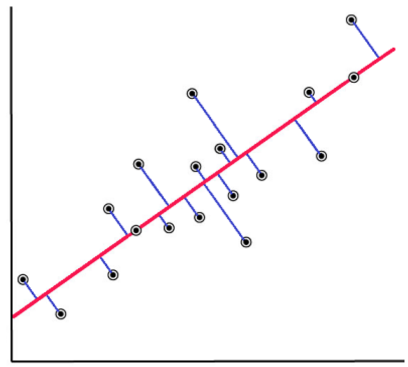
For example, if we multiply the measurement from the problem above (3.64 cm) by the relative error that we just calculated (0.0604), we should get the original absolute error (0.22 cm). Indeed:

$$3.64\text{ cm} \times 0.0604 = 0.22\text{ cm}$$

Use this space for summary and/or additional notes:

Standard Deviation

The standard deviation of a set of measurements is the average of how far each one is from the expected value. For example, suppose you had the following data set, represented by a set of data points and a best-fit line:



If you draw a perpendicular line from each data point to the best-fit line, each perpendicular line would represent how far that data point deviates from the best-fit line. The average of all these deviations is the ***standard deviation***, represented by the variable σ.

There are different types of standard deviation. The formula for the sample standard deviation (which you would use if your data are assumed to be a representative sample taken from a larger data set) is:

$$\sigma_s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

Calculators that include statistics functions and spreadsheet programs have the standard deviation function built in, so you will probably never need to calculate a standard deviation by hand.

Uncertainty, in the kinds of experiments you would do in physics, is usually the standard deviation of the mean (*i.e.,* the standard deviation assuming your data are *all* of the data). To calculate the standard deviation of the mean, divide the sample standard deviation by the square root of the number of data points ($\sqrt{n}$).

$$u = \sigma_m = \frac{\sigma_s}{\sqrt{n}}$$

Use this space for summary and/or additional notes:

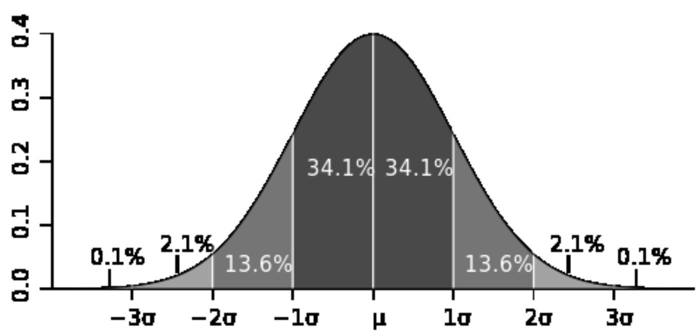
If we had a large number of data points, we could plot a graph of how many data points were how close to the expected value. We would expect that most of the time, the result would be close to the expected value. As you consider results farther and farther from the expected value, you would expect to see these happen less and less often.

This is called a Gaussian or "normalized" distribution. Approximately 68.2% of the measurements would be expected to fall within one standard deviation (±*u*) of the expected value. *E.g.,* if we report a measurement of (22.336 ± 0.030) cm, that means:

- The mean value is $\bar{x} = 22.336\,\text{cm}$
- The standard deviation is $\sigma = 0.030$. This means it is 68% likely that any single measurement we make would fall between 22.306 cm (which equals 22.336 − 0.030) and 22.366 cm (which equals 22.336 + 0.030).

Two standard deviations (±2*u*) should include 95.4% of the measurements, and three standard deviations (±3*u*) should include 99.6% of the measurements.

A graph of the Gaussian distribution looks like a bell, and is often called a "bell curve".



Note that with a very large number of data points, the likelihood that the *average* of the measurements is close to the expected value is much higher than the likelihood that any individual measurement is close.

Use this space for summary and/or additional notes:

Calculating the Uncertainty of a Set of Measurements

When you have measurements of multiple separately-generated data points, the uncertainty is calculated using statistics, so that some specific percentage of the measurements will fall within the average, plus or minus the uncertainty.

Ten or More Independent Measurements

If you have a large enough set of independent measurements (at least 10), then the uncertainty is the standard deviation of the mean. (Independent measurements means you set up the situation that generated the data point on separate occasions. *E.g.,* if you were measuring the length of the dashes that separate lanes on a highway, independent measurements would mean measuring different lines that were likely generated by different line painting apparatus. Measuring the same line ten times would not be considered independent measurements.) That formula is:

$$u = \sigma_m = \frac{\sigma_s}{\sqrt{n}}$$

where:

u = standard uncertainty

σ_m = standard deviation of the mean

σ_s = sample standard deviation $= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$

n = number of measurements

If the variable x represents the measured quantity and $\bar{x}$ represents the arithmetic mean (average) value, you would express your result as:

$$\text{reported value} = \bar{x} \pm u = \bar{x} \pm \frac{\sigma_s}{\sqrt{n}}$$

While this would be the correct formula to use when possible, often we have too few data points (small values of n), which causes the calculated standard deviation to predict a much larger uncertainty than we probably actually have.

Use this space for summary and/or additional notes:

Approximation for Fewer than Ten Independent Measurements

If you have only a few independent measurements (fewer than 10), then you have too few data points to accurately calculate the population standard deviation. In this case, we can estimate of the standard uncertainty by using the formula:

$$u \approx \frac{r}{\sqrt{3}}$$

where:

- u = uncertainty
- r = range (the difference between the largest and smallest measurement)

If the variable x represents the measured quantity, you would express your result as:

$$\text{reported value} = \bar{x} \pm \frac{r}{\sqrt{3}}$$

Note that we are treating $\sqrt{3}$ as a constant. Whenever you have more than one but fewer than ten data points, find the range and divide it by $\sqrt{3}$ to get the estimated uncertainty.

Example:

Suppose you measured a mass on a balance and the reading drifted between 3.46 g and 3.58 g:

$$\bar{x} = \frac{3.46 + 3.58}{2} = 3.52$$

$$r = 3.58 - 3.46 = 0.12$$

$$u \approx \frac{r}{\sqrt{3}} \approx \frac{0.12}{1.732} \approx 0.07$$

You would record the balance reading as (3.52 ± 0.07) g.

Use this space for summary and/or additional notes:

Uncertainty of a Single Measurement

If you are measuring a quantity that is not changing (such as the mass or length of an object), you can measure it as many times as you like and you should get exactly the same value every time you measure it. This means you have only one data point.

When you have only one data point, you can often assume that the standard uncertainty is the limit of how precisely you can measure it (including any estimated digits). This will be your best educated guess, based on how closely you think you actually measured the quantity. This means you need to take measurements as carefully and precisely as possible, because *every careless measurement needlessly increases the uncertainty of the result*.

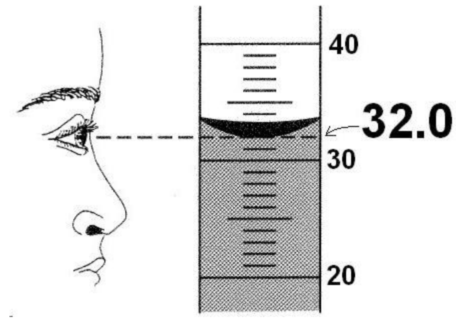
Digital Measurements

For digital equipment, if the reading is *stable* (not changing), look up the published precision of the instrument in its user's manual. (For example, many balances used in high schools have a readability of 0.01 g but are only precise to within ± 0.02 g.) If there is no published value (or the manual is not available), assume the uncertainty is ± 1 in the last digit.

Use this space for summary and/or additional notes:

Analog Measurements

When making analog measurements, always estimate to one extra digit beyond the finest markings on the equipment. For example, in the diagram below, the graduated cylinder is marked in 1 mL increments. When measuring volume in this graduated cylinder, you would estimate and write down the volume to the nearest 0.1 mL, as shown:



In the above experiment, you should record the volume as 32.0 ± 0.1 mL. It would be inadequate to write the volume as 32 mL. (Note that the zero at the end of the reading of 32.0 mL is not extra. It is necessary because *you measured the volume to the nearest 0.1 mL and not to the nearest 1 mL.*)

When estimating, you can generally assume that the estimated digit has an uncertainty of ±1. This means the uncertainty of the measurement is usually $\pm\frac{1}{10}$ of the finest markings on the equipment. Here are some examples:

Equipment	**Markings**	**Estimate To**	**Uncertainty**
triple-beam balance	0.01 g	0.001 g	± 0.001 g
ruler	1 mm	0.1 mm	± 0.1 mm
100 mL graduated cylinder	1 mL	0.1 mL	± 0.1 mL
thermometer	1°C	0.1°C	± 0.1°C

Use this space for summary and/or additional notes:

Propagating Uncertainty in Calculations

When you perform calculations using numbers that have uncertainty, you need to propagate the uncertainty through the calculation.

Crank Three Times

The simplest way of doing this is the "crank three times" method. The "crank three times" method involves:

1. Perform the calculation using the actual numbers. This gives the result (the part before the ± symbol).
2. Perform the calculation again, using the end of the range for each value that would result in the smallest result. (Note that with fractions, this means you need to subtract the uncertainty for values in the numerator and add the uncertainty for values in the denominator.) This gives the lower limit of the range.
3. Perform the calculation again using the end of the range for each value that would result in the largest result. This gives the upper limit of the range.
4. If you have fewer than ten data points, use the approximation that the uncertainty = $u \approx \frac{r}{\sqrt{3}}$, where *r* is the range.

The advantage to "crank three times" is that it's easy to understand and you are therefore less likely to make a mistake. The disadvantage is that it can become unwieldy when you have multi-step calculations.

Use this space for summary and/or additional notes:

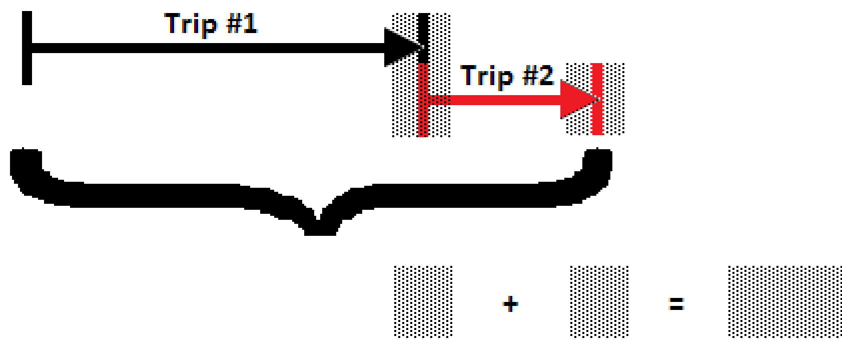
Error Propagation

This method, which we will use throughout the year, requires applying a set of rules based on the formulas you use for the calculations. This is probably the method your college professors will expect you to use in lab experiments.

Addition & Subtraction

If the calculation involves ***addition*** or ***subtraction***, add the *absolute* errors.

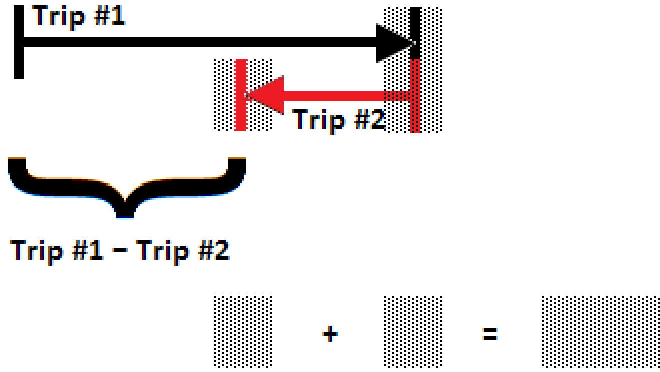
Imagine you walked for a distance and measured it. That measurement has some uncertainty. Then imagine that you started from where you stopped and walked a second distance and measured it. The second measurement also has uncertainty. The total distance is the distance for Trip #1 + Trip #2.



Because there is uncertainty in the distance of Trip #1 and also uncertainty in the distance of Trip #2, it is easy to see that the total uncertainty when the two trips are added together is the sum of the two uncertainties.

Use this space for summary and/or additional notes:

Now imagine that you walked for a distance and measured it, but then you turned around and walked back toward your starting point for a second distance and measured that. Again, both measurements have uncertainty.



Notice that, even though the distances are subtracted to get the answer, the uncertainties still accumulate. As before, the uncertainty in where Trip #1 ended becomes the uncertainty in where Trip #2 started. There is also uncertainty in where Trip #2 ended, so again, the total uncertainty is the *sum* of the two uncertainties.

For a numeric example, consider the problem:

$$(8.45 \pm 0.15)\ \text{cm} - (5.43 \pm 0.12)\ \text{cm}$$

Rewriting in column format:

$$\begin{array}{r} (8.45\ \pm\ 0.15)\ \text{cm} \\ \underline{-\,(5.43\ \pm\ 0.12)\ \text{cm}} \\ \boxed{(3.02\ \pm\ 0.27)\ \text{cm}} \end{array}$$

Notice that even though we had to subtract to find the answer, we had to add the uncertainties.

Use this space for summary and/or additional notes:

Multiplication and Division

If the calculation involves ***multiplication or division***, add the *relative* errors to get the total relative error.

Then, as we saw before, we can multiply the total relative error by the result to get the absolute uncertainty.

For example, if we have the problem (2.50 ± 0.15) kg × $(0.30 \pm 0.06)\,\frac{m}{s^2}$), we would do the following:

The result is $2.50\,\text{kg} \times 0.30\,\frac{m}{s^2} = 0.75\,\frac{kg \cdot m}{s^2}$

For the error analysis:

The relative error of (2.50 ± 0.15) kg is $\frac{0.15}{2.50} = 0.06$

The relative error of $(0.30 \pm 0.06)\,\frac{m}{s^2}$ is $\frac{0.06}{0.30} = 0.20$

The total relative error is 0.06 + 0.20 = 0.26.

The total relative error is what we multiply the result by to get its (absolute) uncertainty:

$0.75\,\frac{kg \cdot m}{s^2} \times 0.26 = \pm\, 0.195\,\frac{kg \cdot m}{s^2}$

The answer with its uncertainty is $(0.75 \pm 0.195)\,\frac{kg \cdot m}{s^2}$

We should make sure our answer and uncertainty are reported to the same place value, so we need to round the uncertainty to get our final answer of $(0.75 \pm 0.20)\,\frac{kg \cdot m}{s^2}$.

Often, you can use the uncertainty to decide where to round your answer. For the kinds of experiments you will do in physics class, it is usually sufficient to show the uncertainty to one or two significant figures, and then round the answer to the same place value.

Use this space for summary and/or additional notes:

Exponents

Calculations that involve ***exponents*** use the same rule as for multiplication and division. If you think of exponents as multiplying a number by itself the indicated number of times, it means you would need to add the relative error of that number that many times.

In other words, when a value is raised to an exponent, multiply its relative error by the exponent.

For example, consider the problem:

$$(0.5) \times (12 \pm 0.8)\ \text{kg} \times \left((6.0 \pm 0.3)\ \tfrac{\text{m}}{\text{s}}\right)^2$$

The result of the calculation is: $(0.5)(12)(6.0)^2 = 216\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}$

The relative errors of the two measurements are:

$$\frac{0.8}{12} = 0.0667 \quad \text{and} \quad \frac{0.3}{6.0} = 0.05$$

Because the $6.0\ \frac{\text{m}}{\text{s}}$ is squared in the calculation, we need to multiply its relative error by two (the exponent). This gives a total relative error of:

$0.0667 + (0.05 \times 2) = 0.1667$

Now multiply the total relative error by the result to get the uncertainty:

$$216\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2} \times 0.1667 = 36\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}$$

Our answer is therefore $\boxed{(216 \pm 36)\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}}$.

Rounding

Just as you should never introduce uncertainty by measuring carelessly, you should also never introduce uncertainty by rounding carelessly. You should always keep all digits in the calculator and round *only* at the end.

However, if for some reason you need to off an intermediate value, estimate how many significant figures your answer will have, and carry at least two additional significant figures (guard digits) through the calculation. When in doubt, keep an extra digit or two. It is much better to have a couple of extra digits than to lose significance because of rounding.

Use this space for summary and/or additional notes:

Summary of Uncertainty Calculations

Uncertainty of a Single Quantity

Measured Once

Make your best educated guess of the uncertainty based on how precisely you were able to measure the quantity and the uncertainty of the instrument(s) that you used.

Measured Multiple Times (Independently)

- If you have a lot of data points, the uncertainty is the standard deviation of the mean, which you can get from a calculator that has statistics functions.
- If you have few data points, use the approximation $u \approx \frac{r}{\sqrt{3}}$.

Uncertainty of a Calculated Value

- For addition & subtraction, add the uncertainties of each of the measurements. The sum is the uncertainty of the result.
- For multiplication, division and exponents:
    1. Find the relative error of each measurement.
       $$\text{R.E.} = \frac{\text{uncertainty } (\pm)}{\text{measured value}}$$
    2. Multiply the relative error by the exponent (if any).
    3. Add each of the relative errors to find the total relative error.
    4. The absolute uncertainty (±) is the result times the total R.E.

Use this space for summary and/or additional notes:

Homework Problems

Because the answers are provided, you must show sufficient work in order to receive credit.

1. A pitching machine pitched five baseballs a distance of 18.44 m in the following times: 0.523 s, 0.506 s, 0.541 s, 0.577 s, and 0.518 s.

   i. What was the average time that it took for each baseball to get from the pitching machine to home plate?

   Answer: 0.533 s

   ii. Assuming each time interval was measured within ±0.001 s, use the "fewer than ten data points estimate" to calculate the uncertainty (±) in the time it took for the baseballs to travel from the pitching machine to home plate.

   Answer: ±0.041 s

Use this space for summary and/or additional notes:

2. A pitching machine pitched five baseballs a distance of d in the following times: t_1, t_2, t_3, t_4, and t_5.

    i. Derive an expression for the average time that it took for each baseball to get from the pitching machine to home plate. (You may use your work from problem 1 above to guide you through the algebra.)

    Answer: $\bar{t} = \dfrac{t_1 + t_2 + t_3 + t_4 + t_5}{5} = \dfrac{\sum_{i=1}^{n} t_i}{n}$

    ii. Assuming each time interval was measured within $\pm t_u$, use the "fewer than ten data points estimate" to derive an expression for the uncertainty (±) in the time it took for the baseballs to travel from the pitching machine to home plate.

    Answer: $t_u = \dfrac{t_{\max} - t_{\min}}{\sqrt{3}}$

3. After school, you drove a friend home and then went back to your house. According to your car's odometer, you drove 3.4 miles to your friend's house (going past your house on the way). Then you drove 1.2 miles back to your house. If the uncertainty in your car's odometer reading is 0.1 mile, how far is it from school directly to your house (including the uncertainty)?

    Answer: (2.2 ± 0.2) mi.

Use this space for summary and/or additional notes:

4. A rock that has a mass of $(8.15 \pm 0.25)\,\text{kg}$ is sitting on the top of a cliff that is $(27.3 \pm 1.1)\,\text{m}$ high. What is the gravitational potential energy of the rock (including the uncertainty)? The formula for gravitational potential energy is $U_g = mgh$, and g is the acceleration due to gravity on Earth, which is equal to $10\,\frac{\text{m}}{\text{s}^2}$.

   Answer: $(2\,225 \pm 158)\,\text{J}$

5. You drive West on the Mass Pike, from Route 128 to the New York state border, a distance of 127 miles. The EZ Pass transponder determines that your car took 1 hour and 54 minutes to complete the trip, and you received a ticket in the mail for driving $66.8\,\frac{\text{mi.}}{\text{hr.}}$ in a $65\,\frac{\text{mi.}}{\text{hr.}}$ zone. The uncertainty in the distance is ± 1 mile and the uncertainty in the time is ± 5 seconds. Can you fight the ticket and win? (You can win if you prove that because of the uncertainty, your speed *could* have been less than $65\,\frac{\text{mi.}}{\text{hr.}}$.)

   Answer: No. Your average speed is $(66.8 \pm 0.575)\,\frac{\text{mi.}}{\text{hr.}}$. Subtracting the uncertainty, the minimum that your speed could have been is $66.8 - 0.575 = 66.2\,\frac{\text{mi.}}{\text{hr.}}$.

6. Solve the following expression and round off the answer appropriately, according to the rules for significant figures:

$$23.5 + 0.87 \times 6.02 - 105$$

   Answer: $-7\overline{6}.2626 = -76$

Use this space for summary and/or additional notes: